This equation is exact. We have, therefore

$$\frac{\partial f}{\partial x} = \frac{3y + 2x}{x(x + y)} = \frac{3}{x} - \frac{1}{x + y} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{2y + x}{y(x + y)} = -\left[\frac{1}{y} + \frac{1}{x + y}\right].$$

Integrating the first equation with respect to x (keeping y as a constant), we get

$$f(x, y) = \int \left(\frac{3}{x} - \frac{1}{x + y}\right) dx + g(y) = 3 \ln |x| - \ln |x + y| + g(y).$$

Substituting in the second equation, we obtain

$$\frac{\partial f}{\partial y} = -\frac{1}{x + y} + g'(y) = -\frac{1}{y} - \frac{1}{x + y} \quad \text{or} \quad g'(y) = -\frac{1}{y}.$$

Integrating, we obtain $g(y) = -\ln | y |$.

Hence,

$$f(x, y) = 3 \ln | x | - \ln | x + y | - \ln | y | = \ln c$$

or

$$\frac{x^3}{y(x + y)} = c \quad \text{or} \quad x^3 = cy(x + y)$$

is the required solution, where c is an arbitrary constant.

Exercise 4.4

For the following differential equations, check whether the equation is exact and obtain its general solution

1. $(1 + e^x)\, dx + y\, dy = 0$.
2. $y\, dx + x(1 + y)\, dy = 0$.
3. $2 \cosh x\, dx + \sinh x\, dy = 0$.
4. $\sinh x \cos y\, dx - \cosh x \sin y\, dy = 0$.
5. $(3x^2y + (y/x))\, dx + (x^3 + \ln x)\, dy = 0$.
6. $(xe^{xy} + 2y)\, dy + ye^{xy}\, dx = 0$.
7. $x\, dy + 2y\, dx = xy\, dy$.
8. $x\, dy - y\, dx = e^y(x^2 + y^2)\, dy$.
9. $x\, dx + y\, dy = 2y(x^2 + y^2)\, dy$.
10. $x\, dy - y\, dx + y^2 dx = 0$.
11. $y(1 + 6xy)\, dx + (4y - x)\, dy = 0$.
12. $(2x + e^y)\, dx + xe^y dy = 0$.
13. $(1 + x^2)\, dy + 2xy\, dx = 0$.
14. $2xy\, dx + (x^2 + 1)\, dy = 0$.
15. $(e^{2y} + 1) \cos x\, dx + 2e^{2y} \sin x\, dy = 0$.

Under what conditions, the following differential equations are exact?

16. $xy^3\, dx + ax^2y^2\, dy = 0$.
17. $[f(x) + g(y)]\, dx + [h(x) + k(y)]\, dy = 0$.
18. $(ax + y)\, dx + (kx + by)\, dy = 0$.
19. $(a \sinh x \cos y + b \cosh x \sin y)\, dx + (c \sinh x \cos y + d \cosh x \sin y)\, dy = 0$.

Find the integrating factor and hence solve the following differential equations

20. $(y - 1)\, dx - x\, dy = 0$.
21. $dx + e^{(y-x)}\, dy = 0$.
22. $(x^3 + y^3 + 1)\, dx + xy^2 dy = 0$.
23. $(4y + x^3)\, dx + x\, dy = 0$.

24. $(2y^3xe^y + y^2 + y)\,dx + (y^3x^2e^y - xy - 2x)\,dy = 0.$
25. $y(1 + 3x^3 + 12x^2)\,dx + (x + 4)\,dy = 0.$
26. $y(1 + xy^2)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0.$
27. $(12y + 3y^4 + 4x^3)\,dx + 6x(1 + y^3)\,dy = 0.$
28. $(x^2 + y^2)\,dx - (2xy)\,dy = 0.$
29. $(2x + y)\,dy - (x + 2y)\,dx = 0.$
30. $y^2\,dx + x(x - y)\,dy = 0.$

Solve the following initial value problems.

31. $3x^2y^4dx + 4x^3y^3dy = 0, \quad y(1) = 2.$
32. $(1 + y)\,dy - (1 - x)\,dx = 0, \quad y(1) = 0.$
33. $3y\,dx + 2x\,dy = 0, \quad y(1) = 1.$
34. $2xy\,dx + (x^2 + \pi\cos \pi y)\,dy = 0, \quad y(1) = 1.$
35. $(\cos x + y\sin x)\,dx = (\cos x)\,dy, \quad y(\pi) = 0.$
36. $xe^{x^2+y^2}\,dx + y(1 + e^{x^2+y^2})\,dy = 0, \quad y(0) = 0.$
37. $xy\,dx - (x^2 + y^2)\,dy = 0, \quad y(0) = 1.$
38. $\left(4x^3y^3 + \frac{1}{x}\right)dx + \left(3x^4y^2 - \frac{1}{y}\right)dy = 0, \quad y(1) = 1.$
39. $(x - y\cos x)\,dx - \sin x\,dy = 0, \quad y(\pi/2) = 1.$
40. $(ye^{xy} + 4y^3)\,dx + (xe^{xy} + 12xy^2 - 2y)dy = 0, \quad y(0) = 2.$
41. $(2xy + e^y)\,dx + (x^2 + xe^y)\,dy = 0, \quad y(1) = 1.$
42. $(x^2 + y^2 + x)\,dx + y\,dy = 0, \quad y(1) = 1.$
43. $xy\,dx + (x^2 + 2y^2 + 2)\,dy = 0, \quad y(0) = 1.$
44. Prove that if M and N in $M(x, y)\,dx + N(x, y)\,dy = 0$ satisfy the equation

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} + \frac{k}{x}N$$

then, $F = x^k$ is an integrating factor. Hence, solve $4y\,dx + x\,dy = 0$.

45. Show that $F(x, y)$ is an integrating factor of $M(x, y)\,dx + N(x, y)\,dy = 0$, if and only if

$$\left(M\frac{\partial F}{\partial y} - N\frac{\partial F}{\partial x}\right) + \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)F = 0.$$